Technical Notes

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Freezing and Melting of Materials with Variable Properties and Arbitrary Heat Fluxes

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Nomenclature

c = dimensionless specific heat c'/c'_m
f = dimensionless heat flux, $f(t')/(\alpha'_m/t_0)^{1/2}\rho'_m L$
k = dimensionless conductivity, k'/k'_m
L = latent heat
p = -1 for melting
 = 1 for solidification
s = dimensionless location of melting or fusion line, $s'/(\alpha'_m t_0)^{1/2}$
T = dimensionless temperature, $c'_m T'/L$
t = dimensionless time, t'/t_0
t_0 = arbitrary reference time
T' = dimensional temperature
u = $\int_0^T \rho c \, \mathrm{d}T$
w = $(\partial\alpha/\partial u)_{x=s}$
x = dimensionless distance $x'/(\alpha'_m t_0)^{1/2}$
x' = dimensional distance from the surface
y = variable defined by Eq. (9)
z = variable defined by Eq. (10)
α = dimensionless thermal diffusivity, α'/α'_m
ρ = dimensionless density, ρ'/ρ'_m
σ = $f'(t)s'/\alpha'\rho' L$

Subscripts

0 = condition at the surface
m = condition at the liquid-solid interface

Superscript

 = dimensional variables
(Prime)

Introduction

RECENTLY, Imber and Huang[1] extended an earlier paper by Goodman[2]‡ on the phase change problem in a semi-infinite region with prescribed surface temperatures by including the effect of variable thermal properties. The same type of problem associated with a more practical boundary condition, prescribed heat fluxes at the surface, has not yet been investigated. Complementary to Ref. 1, this Note presents an appropriate solution for phase changes in a semi-infinite region with variable thermal properties and with arbitrary heat fluxes.

Analysis

Consider a semi-infinite region of liquid (or solid) which is initially at its fusion (or melting) temperature, T_m. The surface is suddenly exposed to a heat flux $f(t)$, which causes a phase change. If we assume the thermal properties of the liquid or solid are functions of temperature only, the dimensionless governing equations describing this system may be written as

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) \qquad t>0 \quad 0\leq x\leq s(t) \tag{1}$$

$$T=T_m \quad @ \ t=0 \tag{2}$$

$$k\frac{\partial T}{\partial x}=pf(t) \quad @ x=0 \tag{3}$$

$$k\frac{\partial T}{\partial x}=p\frac{\mathrm{d}s}{\mathrm{d}t} \quad @ \ x=s(t) \tag{4}$$

$$T=T_m \quad @ \ x\geq s(t) \tag{5}$$

where $p=1$ represents the case of solidification and $p=-1$ for melting, so that the heat flux function $f(t)$ is always positive. For convenience, but without losing generality, the melting or solidification temperature T_m is set equal to zero.

Employing the following Goodman transformation

$$u=\int_0^T \rho c \, \mathrm{d}T \tag{6}$$

and assuming a parabolic temperature profile inside the newly formed solid (or liquid) we than apply the heat balance integral method and obtain surprisingly simple expressions involving the surface temperature and the melting line location. They are

$$u_0=\frac{y}{2}-\frac{z}{4} \tag{7}$$

$$6p-\frac{6p}{s}\int_0^t f(t)\,\mathrm{d}t=y-z \tag{8}$$

where

$$y=-pf(t)s/\alpha_0 \tag{9}$$

$$z=\frac{-1+\left[1+4\dfrac{pf(t)s}{\alpha_0}\left(w+\dfrac{1}{p}\right)\right]^{1/2}}{(w+1/p)} \tag{10}$$

$$w=\left.\frac{\mathrm{d}\alpha}{\mathrm{d}u}\right|_{x=s}$$

and

$$\alpha=\alpha'/\alpha'_m$$

As can be seen, the complicated boundary value problem involving a nonlinear differential equation and a nonlinear

Received July 28, 1975; revision received December 12, 1975.
Index categories: Heat Conduction; Liquid and Solid Thermophysical Properties.

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‡Despite its typographical error, Eq. (44) of Ref. 2 has been frequently reproduced or referred to, e.g., see Refs. 3 and 4. The second term of this equation should read

$$-\frac{2(\beta-1)}{\sqrt{\beta}}\ln\left\{\frac{[1+\beta s(2+s)]^{1/2}+(1+s)\sqrt{\beta}}{1+\sqrt{\beta}}\right\}$$

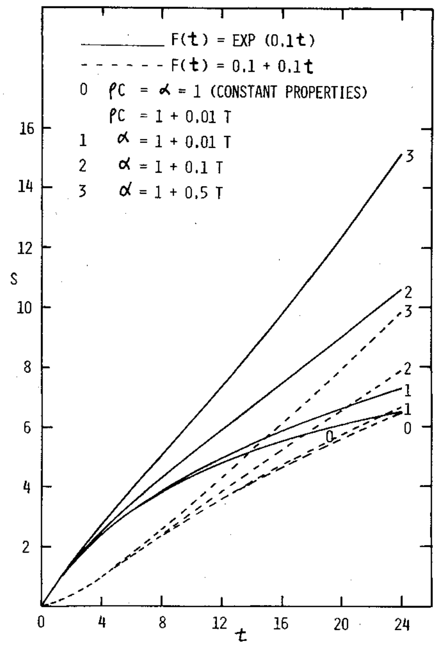

Fig. 1 Dimensionless time-variant melting-line motion ($\rho c = 1 + 0.01T$).

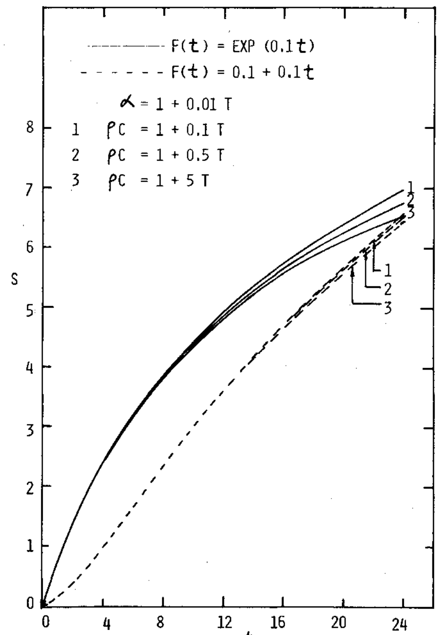

Fig. 3 Dimensionless time-variant melting-line motion ($\alpha = 1 + 0.01\ T$).

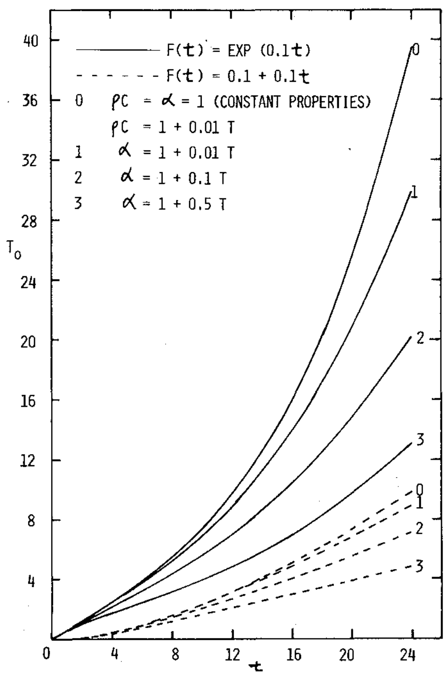

Fig. 2 Dimensionless surface temperature history of the liquid ($\rho c = 1 + 0.01\ T$).

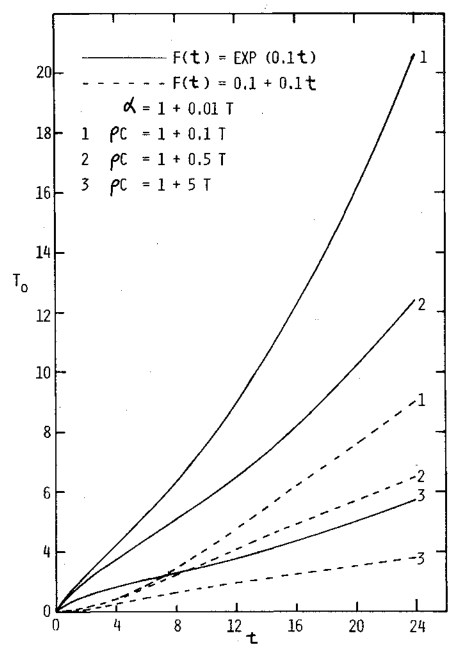

Fig. 4 Dimensionless surface temperature history of the liquid ($\alpha = 0.01\ T$).

boundary condition is reduced to a system of simultaneous algebraic equations represented by Eqs. (7) and (8).

Discussion

We first examine some limiting solutions from the present analysis. For constant thermal properties, i.e., $\rho_0 = 1$, $\alpha_0 = 1$ and $w = 0$, Eq. (8) is reduced to

$$s\cdot f(t) + \sqrt{1+4f(t)\cdot s} = -5 + \frac{6}{s}\int_0^t f(t)\,dt \qquad (11)$$

which agrees with the earlier results of Ref. 2. It is of interest to note that the parameter p disappears in the previous expression. This implies that the formulation for solidification is equally applicable to the case of melting only when the thermal properties are assumed constant. For the case of constant heat flux, $f(t) = f$ Eq. (8) is simplied to

$$\frac{f\cdot s}{\alpha_0} + \frac{-1 + [1 + \frac{4pfs}{\alpha_0}(w + \frac{1}{p})]^{1/2}}{1+pw} = 6(\frac{f\cdot t}{s} - 1) \qquad (12)$$

Note that s is in terms of p in Eqs.(8) and(12).

We now consider some typical examples of melting of solids ($p = 1$) in which the heat fluxes are linear or exponential functions of time and the thermal properties are linear functions of temperature, i.e., $\alpha = 1 + b_1 T$, $\rho c = 1 + b_2 T$ with b_1, and b_2 being constants. Numerical solutions based on Eqs. (7) and (8) are illustrated graphically in Figs. 1-4. The first two figures show the time variant location of the solid-liquid interface and the surface temperature history. The heat capacity for this case is in terms of $\rho c = 1 + 0.01\ T$ and b_1 is a parameter. The solid lines represent the results for which the heat flux is an exponential function of time, and the dash lines correspond to the situation with the heat flux being a linear function of time. The results with constant thermal properties are also included for the purpose of comparison. The same type of curves are presented in Figs. 3-4 with b_1 fixed and b_2 varied. It is seen that the quantity b_1 tends to increase s and decrease T_0 but the increase of b_2 reduces both s and T_0. Furthermore, the variation of b_2 does not affect the melting rate very much, although it influences the surface temperature considerably.

References

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Stability of Laser Heated Flows

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Introduction

THE success of any system designed to convert high power laser radiation into directed kinetic energy depends upon the ability of the system to perform this conversion efficiently. Neither the work on laser propulsion[1-3] nor Hertzberg et al.[4] have *a priori* addressed the fundamental issue of the most efficient process by which a flowing gas can absorb laser radiation and convert it to useful work. Pirri et al.[1,2] relied upon the creation of electrons via breakdown to initiate absorption by inverse Bremsstrahlung and subsequent conversion of the heated gas into kinetic energy to obtain high specific impulse. However, these experiments illustrated *a posteriori* that without some tailoring of the propellant composition, the conversion process yields the initiation of absorption waves in the gas.[2] This results in unsteady flow behavior and low power conversion efficiency.

The stability of radiatively heated flows along with suggestions for acoustic wave amplification by radiative absorption have been examined to some extent by many authors.[5] However, due to the inhomogeneity in the one-dimensional nozzle flow, when the disturbances are introduced, the full eigenvalue problem is rather difficult. Fortunately, much can be learned about wave propagation without solving the full boundary value problem, and information can be extracted from a study of the modified dispersion relation. To obtain the modified dispersion relation in inhomogeneous media the actual disturbance is approximated by harmonic functions over a small distance (small compared with the characteristic lengths associated with the gradients of the flow), and, hence, the analysis is only an accurate description "locally." The procedure is commonly used in problems of this type, e.g., Monsler[6] has examined the instability of radiative transfer between two infinite plates. Here we apply the "local" stability analysis to a given nozzle to obtain contour (or contours) of neutral stability.

Received August 7, 1975; revision received December 3, 1975. This work was sponsored by NASA-Lewis, under Contract NAS3-18528 and NASA Project Manager, Mr. Stephen M. Cohen.

Index category: Hydrodynamics; Nozzle and Channel Flow; Electric and Advanced Space Propulsion.

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Method of Analysis

For simplicity, consider a one-dimensional flow through a nozzle of variable area ratio that is heated by laser radiation. The laser beam enters from the up-stream direction. The governing equations for the quasi-one-dimensional flow without viscous dissipation, diffusion, and thermal conduction but including radiative heat transfer are

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho u A) = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \tag{2}$$

$$\frac{\partial h_s}{\partial t} + u\frac{\partial h_s}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial t} = \frac{1}{\rho A}\frac{\mathrm{d}(IA)}{\mathrm{d}x} \tag{3}$$

where ρ is the density; u is the velocity; A is the area; p is the pressure; h_s is the stagnation enthalpy; and I is the intensity. For the radiative source term, it is assumed that

$$IA = I_i A_i e^{-\tau}$$

where τ is the optical depth defined as[7]

$$\tau = \int_0^x \kappa\,\mathrm{d}x \tag{4}$$

and a generalized absorption coefficient κ is assumed to be of the form

$$\kappa = \kappa_c \rho^n T^m \tag{5}$$

where κ_c is a constant; T is the gas temperature; and n and m are constants characterizing the absorber. All radiation quantities (I, I_i, κ, τ, κ_c) are quasi-monochromatic in the sense of Ref. 8. Finally the equation-of-state, $p = \rho RT$, together with Eqs. (1-3) provide the necessary equations for variables ρ, u, p, and T.

The governing equations for the steady-state solutions are the same as the previous equations with the time derivatives set equal to zero. Substituting Eq. (4) into Eq. (3) and integrating the resulting steady-state energy equation yields

$$h_s(\tau) = 1 + \Gamma(1 - e^{-\tau})$$

where the stagnation enthalpy is normalized by its initial value h_{si} and the important parameter Γ, the ratio of laser power to the initial total energy flux of the flow, is defined as

$$\Gamma = (IP/(\rho u A H_{si})$$

where IP is the laser power.

A general method of solution for the steady-state equations may be employed.[9] The governing equations can be combined to yield a relationship which governs the Mach number variation through the nozzle.

$$\frac{\mathrm{d}M^2}{\mathrm{d}\tau} = \frac{1}{1-M^2}\left[1 + \frac{(\gamma-1)}{2}M^2\right]M^2\left[(1+\gamma M^2)\times\frac{1}{h_s}\frac{\mathrm{d}h_s}{\mathrm{d}\tau} - \frac{2}{A}\frac{\mathrm{d}A}{\mathrm{d}\tau}\right] \tag{6}$$